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Francis Conrad, Michel Pierre

UNITE DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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Francis CONRAD
Michel PIERRE

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Abstract

For an abstract evolution equation of the form $u_{tt} + Au + \partial\psi(u_t) \ni 0$, general conditions on the "unbounded" feedback are given, that ensure strong asymptotic stability. Essentially the directions determined by the convex of the minima of the functional ψ should not intersect the eigenspaces of A. Equivalently, the feedback on the velocity must dissipate enough energy, in the sense that the kernel of the form $< \partial\psi(.), . >$ is not larger than the kernel of a "strategic" observation operator, for the uncontrolled system. The particular case where the control operator is the dual of the observation operator is specifically considered : the condition then corresponds to more classical rank conditions on the observation operator. The present framework applies to boundary or interior, distributed or pointwise, controls. The analysis is also able to handle "unilateral controls". Several examples, including wave, beam and plate equations, possibly with interior control on thin sets, are considered.

Stabilisation de Problèmes d'Evolution du Second Ordre par des Feedbacks Non Linéaires Non Bornés

Résumé

Pour un problème d'évolution abstrait de la forme $u_{tt} + Au + \partial\psi(u_t) \ni 0$, on donne des conditions générales sur le feedback "non borné" pour assurer la stabilité asymptotique forte. Essentiellement les directions déterminées par le convexe des minima de ψ ne doivent pas être des directions propres de l'opérateur A. De façon équivalente, il faut que le bouclage sur la vitesse soit suffisamment dissipatif, en ce sens que le noyau de la forme $< \partial\psi(.), . >$ ne doit pas être plus gros que le noyau d'un observateur "stratégique", pour le système non contrôlé. Le cas particulier où l'opérateur de contrôle est l'adjoint de l'opérateur d'observation est étudié: la condition se ramène alors à des conditions plus classiques de rang sur l'opérateur d'observation. Le cadre proposé ici englobe le cas de contrôles frontière ou intérieurs, distribués ou ponctuels, ainsi que des contrôles unilatéraux. Divers exemples concernant les équations des ondes, des poutres ou des plaques, éventuellement avec des contrôles sur des ensembles intérieurs "fins", sont proposés.

(*) Université de Nancy I, URA-CNRS 0750 et Projet NUMATH, INRIA-Lorraine
B.P. 239, 54506 Vandoeuvre-lès-Nancy, France.

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1. Abstract framework - Wellposedness

Let H be a Hilbert space, and let A be a linear operator with dense domain D(A). We assume A is self-adjoint, coercive on H, and we define $V = D(A^{1/2})$, equipped with the scalar product

(1) $\quad <u,v>_{VxV} = (A^{1/2}u, A^{1/2}v)_{HxH} = <\tilde{A}\, u,v>_{V'xV}$

where $\tilde{A} \in \mathscr{L}(V,V')$ is defined by the bilinear form $< , >_{VxV}$ and extends A. As usual, we identify H with its dual. Then $V \hookrightarrow H \hookrightarrow V'$, with the following consistency relation

(2) $\quad \forall\, h \in H, \qquad \forall\, v \in V, \quad <h,v>_{V'xV} = (h,v)_{HxH}$.

Let be given a proper, convex, lower semi-continuous (l.s.c.) function

(3) $\quad \psi : V \to]-\infty, \infty], \quad \psi \not\equiv +\infty$

with effective domain $D(\psi) = \{v \in V \,;\, \psi(v) < \infty\}$. We consider the sub-differential $\partial\psi$ of ψ defined by

(4) $\quad \forall\, u \in V, \partial\psi(u) = \{f \in V' \,;\, \psi(u+v) - \psi(u) \geq <f,v>_{V'xV}, \forall\, v \in V\}$,
with $D(\partial\psi) = \{u \in V \,;\, \partial\psi(u) \neq \emptyset\}$.

It is known that $\partial\psi$ is a maximal monotone graph from V to V', and that $D(\partial\psi) \subset D(\psi)$, with a dense inclusion [BA-PRE].

Next, on the space VxH equipped with the natural product Hilbert structure, we define the nonlinear operator B by

(5) $\quad D(B) = \{(v,h) \in VxH \,;\, h \in D(\partial\psi) \,;\, \exists\, f \in \partial\psi(h) \text{ such that } \tilde{A}\, v + f \in H\}$
and if $(v,h) \in D(B)$

(6) $\quad B(v,h) = \{(-h, \ \tilde{A}\, v + f) \,;\, \forall\, f \in \partial\psi(h) \text{ such that } \tilde{A}\, v + f \in H\}$.

Equivalently, in terms of graphs considered as subsets of VxH

(7) $\quad B = \{(v,h) \times (-h, [\, \tilde{A}\, v + \partial\psi(h)] \cap H) \,;\, (v,h) \in VxH\}$.

We now prove that we have an adequate general framework to define a second order evolution equation.

Proposition 1. B is maximal monotone on VxH.

Proof.

(i) The monotonicity is easy to establish. Let $(v,h), (\tilde{v},\tilde{h}) \in D(B)$, $f \in \partial\psi(h)$, $\tilde{f} \in \partial\psi(\tilde{h})$, such that $\tilde{A}\, v + f$ and $\tilde{A}\, \tilde{v} + \tilde{f} \in H$. Then (see (1) and (2))

$$< -h + \tilde{h}, v - \tilde{v} >_{VxV} + (\tilde{A}\, v + f - \tilde{A}\, \tilde{v} - \tilde{f},\ h - \tilde{h})_{HxH} =$$
$$< \tilde{A}\, (v - \tilde{v}), \tilde{h} - h >_{V'xV} + < \tilde{A}\, v + f - \tilde{A}\, \tilde{v} - \tilde{f},\ h - \tilde{h} >_{V'xV} =$$

(8) $\quad < f - \tilde{f}, h - \tilde{h} >_{V'xV} \geq 0$, since $\partial\psi$ is monotone on VxV'.

(ii) We prove that I + B is onto on VxH : for (F,G) $\in$ VxH, we have to solve the system

$$(9) \quad \begin{cases} (v,h) \in D(B) \\ v - h = F \\ h + \tilde{A}\, v + f = G \ , \ f \in \partial\psi(h) \quad (\text{thus } \tilde{A}\, v + f \in H) \end{cases}$$

which is equivalent to the system

$$(10) \quad \begin{cases} v \in V \ , \ h \in D(\partial\psi) \subset V & \text{(a)} \\ v = F + h & \text{(b)} \\ h + \tilde{A}h + f = G - \tilde{A}F \ , \ f \in \partial\psi(h) & \text{(c)} \end{cases}$$

(observe that (9) $\Rightarrow$ (10) obviously and that, if (10) holds, then $\tilde{A}\, v + f = \tilde{A}h + f + \tilde{A}F = G - h \in H$).

Essentially, we have to prove that $(10)_c$, holds i.e. that the operator $I + \tilde{A} + \partial\psi$ from V to V' is onto. But this is easy to do, and more or less standard. We recall briefly the procedure. Let

$$J(\theta) = \frac{1}{2}\ (\theta,\theta)_{HxH} + \frac{1}{2} < \theta,\theta >_{VxV} + \psi(\theta) - < G - \tilde{A}F,\theta >_{V'xV}$$

which is a convex, l.s.c. functional on V. Since ψ is convex, l.s.c. and proper, it is bounded below by an affine function, and hence, $\lim J(\theta) = +\infty$, as $|\theta|_V \to +\infty$. Therefore, J admits at least a minimum at $h \in V$. Then

$$J(h) \leq J(h + t\,(\theta - h)),\ \forall\ \theta \in V,\ \forall\ t \in [0,1].$$

Writing this inequality, using the convexity of ψ,then dividing by t and finally letting $t \searrow 0_+$, we get

$$< G - \tilde{A}F,\theta - h >_{V'V} \ \leq (h, \theta - h)_{HxH} + < h, \theta - h >_{VxV} + \psi(\theta) - \psi(h),\ \forall\ \theta \in V.$$

Since $< h, \theta - h >_{VxV} \ = < \tilde{A}h, \theta - h >_{V'xV}$ and $(h, \theta - h)_{HxH} = < h, \theta - h >_{V'xV}$, we deduce that $G - \tilde{A}F - h - \tilde{A}h \in \partial\psi(h)$, which is just $(10)_c$. ■

We denote by $S^B(t)$ the nonlinear semi-group of contractions on $\overline{D(B)}$ generated by B. As a consequence of the general theory [BRE], we have the following properties, where $(u(t),v(t)) = S^B(t)\,(u_0, u_1)$:

(11) $\quad \forall\ (u_0, u_1) \in \overline{D(B)}$, $t \to (u(t),v(t)) \in \mathscr{C}([0,\infty[;\ VxH)$

and, for any $(u_0, u_1) \in D(B)$

(12) $\quad \forall\ t \geq 0,\ (u(t), v(t)) \in D(B)$

(13) $\quad t \to (u(t), v(t)) \in VxH$ is Lipschitz continuous and a.e. differentiable on $[0,\infty[$

(14) $\quad v(t) = u_t(t),\ u_{tt}(t) + \tilde{A}\, u(t) + f(t) = 0,\ f(t) \in \partial\psi(v(t))$, a.e. $t \geq 0$.

Moreover, for any $(u_0, u_1) \in D(B)$

(15) $\quad t \rightarrow (-v(t), [\tilde{A}\, u(t) + \partial\psi(v(t))]^o) \in VxH$ is right-continuous everywhere, where $[\ ]^o$ denotes the minimal section.

(16) $\quad \|(-v(t), [\tilde{A}\, u(t) + \partial\psi(v(t))]^o)\|_{VxH} \leq \|(-u_1, [\tilde{A}\, u_0 + \partial\psi(u_1)]^o)\|_{VxH},\ \forall t \geq 0.$

Note that in particular, if $(u_0, u_1) \in D(B)$, then $u \in W^{2,\infty}(0,\infty;H) \cap W^{1,\infty}(0,\infty;V)$.

For any $(u_0, u_1) \in \overline{D(B)}$, we define

(17) $\quad E(t, u_0, u_1) = \frac{1}{2}\, |u(t)|_V^2 + \frac{1}{2}\, |v(t)|_H^2.$

Proposition 2. (i) Assume $(u_0, u_1) \in D(B)$, then $\forall\, 0 \leq s \leq t$

(18) $\quad E(t, u_0, u_1) - E(s, u_0, u_1) = - \int_s^t < f(\sigma), v(\sigma) >_{V'xV}\, d\sigma,$

where $f(\sigma) \in \partial\psi(v(\sigma))$ a.e. σ satisfies (14).

(ii) Assume $0 \in \partial\psi(0)$; then $\forall\, (u_0, u_1) \in \overline{D(B)}$, $t \rightarrow E(t, u_0, u_1)$ is nonincreasing.

Proof.

(i) Let $(u_0, u_1) \in D(B)$; by the regularity property (13), we have, a.e. t :

$$\frac{d}{dt} E(t, u_0, u_1) = < u_t(t), u(t) >_{VxV} + (v_t(t), v(t))_{HxH}$$
$$= < \tilde{A}\, u(t), u_t(t) >_{V'xV} + (- \tilde{A}\, u(t) - f(t), v(t))_{HxH},$$

where $f(t) \in \partial\psi(v(t))$ satisfies (14)

$$= < \tilde{A}\, u(t), u_t(t) >_{V'xV} - < \tilde{A}\, u(t) + f(t), v(t) >_{V'xV} \text{ by (2)}$$
$$= - < f(t), v(t) >_{V'xV}.$$

Then (18) follows by integration.

(ii) Since $\partial\psi$ is monotone and $0 \in \partial\psi(0)$, we have $<f(\sigma), v(\sigma)>_{V'xV} \geq 0$. Thus $t \rightarrow E(t, u_0, u_1)$ is nonincreasing for $(u_0, u_1) \in D(B)$. By density, and continuity of $E(t,.,.)$ on VxH, the property is true for $(u_0, u_1) \in \overline{D(B)}$. ■

Remark 1. Let U be a Hilbert space, $\phi : U \rightarrow]-\infty,\infty]$ be proper, convex and l.s.c., and $C \in \mathscr{L}(V,U)$. If $R(C) \cap D(\phi) \neq \emptyset$ (for instance if C is surjective), then $\psi = \phi \circ C$ is suitable. It has been proved in [LA] (see also Appendix 1) that if C is surjective, then $\partial\psi = C^*\partial\phi C$ and that the equation

$u_{tt} + \tilde{A}u + C^*\partial\phi C\, u_t \ni 0$

defines a nonlinear semi-group of contractions on a closed subset of VxH. This situation corresponds to the case where the control operator C* is the adjoint of the observation operator C, with a nonlinear interaction between observation and control described by $\partial\psi$. Therefore, we have proved a result which includes a familiar framework in control theory.

Remark 2. The fact that $\partial\psi$ acts from V to V' and not necessarily from H to H, is adequate to describe boundary or point feedback between observation and control in P.D.E's (when $\psi = \varphi\circ C$).

Remark 3. Even in the restricted case where $\psi = \varphi\circ C$ and C is surjective, it may be as convenient to keep the formulation with ψ, and to compute $\partial\psi$ directly, without using $C^*\partial\varphi C$. This will be observed later on, in the applications (Section 3).

2. Strong asymptotic stability

We now study the asymptotic behaviour of the semi-group $S^B(t)$. The main tool will be the invariance principle of LASALLE. As usual, some compactness has to be assumed. We suppose that $0 \in \partial\psi(0)$, thus after a normalization

(19) $\quad \min_{v \in V} \psi(v) = \psi(0) = 0.$

We denote by K_ψ the closed convex set where ψ attains its minimum

(20) $\quad K_\psi = \{v \in V;\ \psi(v) = 0\}.$

Moreover, we assume B has compact resolvent (for conditions ensuring the compactness, see Appendix 2). It follows from this fact and from (19) that the trajectories of $S^B(t)$ are relatively compact in VxH, and that for any $(u_0, u_1) \in \overline{D(B)}$, the ω-limit set $\omega(u_0, u_1)$ is a non-empty closed set [DA-SLE].

Proposition 3. Assume (19) and B has compact resolvent. Then

(21) *. If $(u_0, u_1) \in D(B)$, then $\omega(u_0, u_1) \subset D(B)$*

. $\omega(u_0, u_1)$ is invariant under $S^B(t)$, and the restriction of $S^B(t)$ on $\omega(u_0, u_1)$ is an isometry for the VxH-norm.

. Let $(u_0, u_1) \in D(B)$, $(w_0, w_1) \in \omega(u_0, u_1)$, and $(w(t), w_t(t)) = S^B(t)(w_0, w_1)$.

Then

(22) $\quad 0 = \langle f(t), w_t(t)\rangle_{V'xV},$ *where $f(t) \in \partial\psi(w_t(t))$ a.e. t, is defined as in (14).*

. If moreover, ψ satisfies the following property

(23) $\quad \partial\psi(0) = \{0\}$

then

(24) $(w_0, w_1) \in D(B_0)$, *and* $(w(t), w_t(t)) = S^{B_0}(t)\,(w_0, w_1)$,

where B_0 *is the operator B corresponding to* $\partial\psi \equiv \{0\}$, *i.e.*

(25) $D(B_0) = D(A)xV$, *and*

$\forall\,(v,h) \in D(B_0),\ B_0\,(v,h) = (-h, Av)$.

Proof.

The claim (21) follows directly from the maximality of B and the fact that $t \to \|B^0(u(t), u_t(t))\|_{VxH}$ is uniformly bounded, and in fact decreasing along the trajectories (see (16)).

The fact that $S^B(t)$ is an isometry on $\omega(u_0, u_1)$ is a consequence of the invariance principle of LASALLE, since $t \to E(t, u_0, u_1) = \frac{1}{2}\,\|(u(t), v(t))\|^2_{VxH}$ is a Lyapunov function (See Proposition 2 (ii)).

Property (22) follows from formula (18) applied to (w_0, w_1) instead of (u_0, u_1) : $t \to E(t, w_0, w_1)$ is constant, and $< f(\sigma), w_t(\sigma) >_{V'xV}$ is nonnegative a.e..

For properties (24) and (25), we first observe that since $f(t) \in \partial\psi(w_t(t))$ a.e., we get

$$\psi(0) \geq \psi(w_t(t)) - < f(t), w_t(t) >_{V'xV}$$

and thus, from (22) we deduce

$$\psi(w_t(t) \leq \psi(0), \text{ thus } \psi(w_t(t)) = 0 = \psi(0).$$

Let $v \in V$ be arbitrary. Then

$$\psi(v) - \psi(0) = \psi(v) - \psi(w_t(t)) \geq$$

$$< f(t), v - w_t(t) >_{V'xV} = < f(t), v >_{V'xV} \text{ (using (22)).}$$

This implies $f(t) \in \partial\psi(0)$, and by assumption (23) we get $f(t) = 0$ a.e. t. Therefore, w satisfies the equation derived from (14)

$$w_{tt}(t) + \tilde{A}\,w(t) = 0 \text{ a.e. } t \geq 0,$$

thus $\tilde{A}\,w(t) \in H$ a.e. $t \geq 0$. This implies

$$w(t) \in D(A) \text{ a.e. } t \geq 0, \text{ and } \tilde{A}\,w(t) = Aw(t),$$

and also

$$B(w(t), w_t(t))^0 = B_0(w(t), w_t(t)) \text{ a.e. t.}$$

But by (16)

$$\|B_0(w(t), w_t(t))\|_{VxH} = \|B(w(t), w_t(t))^0\|_{VxH} \leq \|B(w_0, w_1)\|_{VxH}, \text{ a.e. t.}$$

Letting t tend to zero, we get $(w_0, w_1) \in D(B_0)$, by maximality of the operator B_0.
Clearly, then $(w(t), w_t(t)) = S^{B_0}(t)\,(w_0, w_1)$. ∎

Next, we come to our main results of the paper, namely a characterization of asymptotic stabilization in terms of ψ (more precisely of K_ψ).

Assume that the resolvent of A is compact. Let $0 < \lambda_1 < \lambda_2 < ... < \lambda_i < ...$ be the distinct eigenvalues of A, with associated eigenspaces F_i of dimension p_i, all finite by the assumption on A. Denote by $\{\varphi_i^j\}_{j=1,...p_i}$ a basis of eigenfunctions of F_i such that the whole system is an orthonormal basis of H (and an orthogonal basis of V).

For the moment, for simplicity, we denote by φ_n, $n \geq 0$ the eigenfunctions of A, with associated eigenvalues $\omega_n^2 > 0$ (not necessarily distinct) and such that

$$|\varphi_n|_H = 1,\ |\varphi_n|_V = |A^{1/2}\varphi_n|_H = \omega_n > 0.$$

It is straightforward that $\frac{1}{\sqrt{2}}\left(\varphi_n/\omega_n\ ,\ \mp\ i\varphi_n\right)$ is a Hilbert basis of eigenfunctions of B_0 in VxH, associated with eigenvalues $\pm\ i\ \omega_n$.

Let $(w_0, w_1) \in \overline{D(B_0)} = V\times H$, $(w(t), w_t(t)) = S^{B_0}(t)\ (w_0, w_1)$. By a straightforward computation we get

(26) $$w(t) = \sum_{\mathbb{N}}[a_n \cos \omega_n t + b_n \sin \omega_n t]\ \varphi_n/\omega_n$$

(27) $$w_t(t) = \sum_{\mathbb{N}}[b_n \cos \omega_n t - a_n \sin \omega_n t]\ \varphi_n$$

where (26) (resp. (27)) converges in V, (resp. H). Moreover

$$a_n = < w_0, \varphi_n/\omega_n >_{V\times V} \in l^2(\mathbb{N}), \text{ since } w_0 \in V \text{ and } |\varphi_n/\omega_n|_V = 1,$$

$$b_n = (w_1, \varphi_n)_{H\times H} \in l^2(\mathbb{N}), \text{ since } w_1 \in H \text{ and } |\ \varphi_n\ |_H = 1.$$

Now, if we assume that $(w_0, w_1) \in D(B_0)$, the convergences in (26) and (27) can be improved.

First, $w_0 \in D(A)$, thus

$$a_n = \frac{1}{\omega_n}\ (A^{1/2}w_0, A^{1/2}\varphi_n)_{H\times H} = \frac{1}{\omega_n}\ (Aw_0, \varphi_n)_{H\times H}$$

which implies

$$\omega_n\ a_n = (Aw_0, \varphi_n)_{H\times H} \in\ l^2(\mathbb{N}), \text{ since } Aw_0 \in H, \text{ and } |\varphi_n|_H = 1.$$

Next $w_1 \in V$, thus

$$b_n = (w_1, \varphi_n)_{H\times H} = \frac{1}{\omega_n^2}\ (w_1, A\varphi_n)_{H\times H} = \frac{1}{\omega_n^2}\ (A^{1/2}w_1, A^{1/2}\varphi_n)_{H\times H}$$

$$= \frac{1}{\omega_n}\ < w_1, \frac{\varphi_n}{\omega_n} >_{V\times V}$$

which implies

$\omega_n b_n = < w_1, \frac{\varphi_n}{\omega_n} >_{V \times V} \in l^2(\mathbb{N})$, since $w_1 \in V$ and $|\varphi_n/\omega_n|_V = 1$.

Since $V = D(A^{1/2}) = \left\{ \sum_{\mathbb{N}} \alpha_n \varphi_n \; ; \; \sum_{\mathbb{N}} \alpha_n^2 \omega_n^2 < \infty \right\}$, we deduce that if $(w_0, w_1) \in D(B_0)$, then, in particular, the series (27) converges in V, uniformly with respect to t.

Returning now to the notations introduced previously, we get

$$(28) \qquad w_t(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{pi} [b_i^j \cos \omega_i t - a_i^j \sin \omega_i t] \, \varphi_i^j,$$

where the series converges in V, uniformly in t.

Theorem 4. Assume A and B have compact resolvent and (19), (23). Then

$$(29) \qquad \forall (u_0, u_1) \in \overline{D(B)}, \qquad \lim_{t \nearrow \infty} S^B(t)\,(u_0, u_1) = 0 \text{ in } V \times H$$

if and only if

$$(30) \qquad \forall i \geq 1, K_\psi \cap (-K_\psi) \cap F_i = \{0\}.$$

Proof.

Sufficiency. Assume (30). By the contraction property of the semi-group $S^B(t)$, it is enough to prove (29) for $(u_0, u_1) \in D(B)$.

Let $(u_0, u_1) \in D(B)$, and $(w_0, w_1) \in \omega(u_0, u_1)$. It is enough to prove that $(w_0, w_1) = 0$.

Let $(w(t), w_t(t)) = S^B(t)(w_0, w_1)$. By Theorem 3, (23) and (24) imply that $(w_0, w_1) \in D(B_0)$, $(w(t), w_t(t)) = S^{B_0}(t)(w_0, w_1)$,
thus formula (28) is valid, and can be written as

$$(31) \qquad w_t(t) = \sum_{i=1}^{\infty} \cos \omega_i t \, \varphi_i + \sin \omega_i t \, \psi_i, \text{ with } \varphi_i, \psi_i \in F_i$$

where the series converges in V, uniformly in t. Thus (31) defines an almost periodic function from $\mathbb{R}$ to V [LE-ZH]. Moreover (see proof of Theorem 3)

$$(32) \qquad w_t(t) \in K_\psi \text{ , a.e. } t \geq 0.$$

From the uniform convergence in (31) we deduce that

$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$(33) \quad \left|w_t(t) - \sum_{i=1}^{N} [\cos \omega_i t\, \varphi_i + \sin \omega_i t\, \psi_i]\right|_V \le \varepsilon,\ \forall t \ge 0$$

and thus, with $\eta = \pm 1$, we deduce

$$\left|\frac{1}{T}\int_0^T \frac{1+\eta \cos \omega_p t}{2} w_t(t) - \sum_{i=1}^{N} \frac{1}{T}\int_0^T \frac{1+\eta \cos\omega_p t}{2}\ (\cos \omega_i t\, \varphi_i + \sin \omega_i t\, \psi_i)\right|_V \le$$

$$(34) \quad \le \left| \frac{1+\eta \cos\omega_p t}{2} \right|_{L^\infty} \varepsilon = \varepsilon.$$

On the other hand, a straightforward computation gives

$$(35) \quad \int_0^T (1+\eta \cos \omega_p t) \cos\omega_i t\, dt = \frac{1}{\omega_i} \sin \omega_i T +$$

$$+\frac{\eta}{2} \begin{cases} \dfrac{1}{\omega_i+\omega_p} \sin (\omega_i+\omega_p)T + \dfrac{1}{\omega_i-\omega_p} \sin (\omega_i-\omega_p)T, & \text{if } i \ne p \\ T + \dfrac{1}{2\omega_p} \sin 2\omega_p T, & \text{if } i = p \end{cases}$$

$$(36) \quad -\int_0^T (1+\eta \cos \omega_p t) \sin\omega_i t\, dt = \frac{1}{\omega_i} \cos \omega_i T +$$

$$+\frac{\eta}{2} \begin{cases} \dfrac{1}{\omega_i+\omega_p} [\cos (\omega_i+\omega_p)T-1] + \dfrac{1}{\omega_i-\omega_p} [\cos (\omega_i-\omega_p)T-1], & \text{if } i \ne p \\ \dfrac{1}{2\omega_p} \cos 2\omega_p T, & \text{if } i = p. \end{cases}$$

From (34), (35) and (36) one obtains for $N > p$

$$(37) \quad \left| \lim_{T \nearrow \infty} \frac{1}{T} \int_0^T \frac{1+\eta \cos \omega_p t}{2} w_t(t)\, dt - \frac{\eta}{4} \varphi_p \right|_V \le \varepsilon.$$

By (32) $w_t(t) \in K_\psi$ a.e. t, and $\dfrac{1+\eta \cos \omega_p t}{2} \in [0,1]$, by the choice of η. Since K_ψ is a closed convex set containing the origin, we get

$$\frac{1 + \eta \cos \omega_p t}{2}\ w_t(t) \in K_\psi \text{ a.e. } t\,,$$

and consequently, with the same argument,

$$(38) \quad \lim_{T \nearrow \infty} \frac{1}{T} \int_0^T \frac{1+\eta \cos \omega_p t}{2} w_t(t) \in K_\psi.$$

From (37) and (38) one deduces

(39) $\pm\frac{1}{4}\ \varphi_p \in K_\psi$.

But, by definition, $\varphi_p \in F_p$. So (30) implies $\varphi_p = 0$. Similarly, multiplying (31) by $\frac{1+\eta \sin \omega_p t}{2}$ instead of $\frac{1+\eta \cos \omega_p t}{2}$ and integrating over (0,T), one gets $\psi_p = 0$, $\forall p$.

Hence $w_t(t) = 0$ for all t, and also $w(t) = 0$ for all t. This proves (29).

Necessity. We argue by contradiction. Suppose that, for some i, there exists

$$v \in K_\psi \cap (-K_\psi) \cap F_i,\ v \neq 0.$$

Then consider $u(t) = -\frac{1}{\omega_i} \cos \omega_i t\ v$, $u_t(t) = \sin \omega_i t\ v$.

It is clear that $u_{tt}(t) = -\omega_i^2\ u(t)$, and thus

(40) $u_{tt} + A\ u = 0.$

So $(u(t), u_t(t)) = S^B0(t)\ (-\frac{1}{\omega_i}\ v, 0)$.

Since $v \in K_\psi \cap (-K_\psi)$, for any $\lambda \in [-1,1]$ $\lambda\ v \in K_\psi$, and thus $\psi(\lambda v) = 0$. Deriving this identity w.r. t λ yields

$$< f,v > = 0 \quad \forall\ f \in \partial\psi(\lambda v), \text{ or else}$$
$$< f, \lambda v > = 0 \quad \forall\ f \in \partial\psi(\lambda v).$$

As in the proof of Proposition 3, one deduces $f \in \partial\psi(0)$, and from condition (23), one obtains

$$\partial\psi(\lambda v) = \{0\}, \qquad \forall\ \lambda \in [-1,1].$$

Applying the result to $\lambda = \sin \omega_i t$ one gets

$$\partial\psi(u_t) = \{0\}.$$

Thus u, which is a solution of (40) is also a solution of

(41) $u_{tt} + A\ u + \partial\psi(u_t) \ni 0,$

So $(u(t), u_t(t)) = S^B(t)\ (-\frac{1}{\omega_i}\ v, 0)$. But $(u(t), u_t(t)) \not\to 0$ as $t \nearrow \infty$, whence the contradiction ■

In applications, the following corollary is often useful.

Corollary 5. Assume $\psi(v) = \varphi(Cv)$, where $C \in \mathscr{L}(V,U)$, with U a Hilbert space and $\varphi : U \to [0,\infty]$ convex, l.s.c., satisfying $\varphi(0) = \min \varphi = 0$, and

(42) $\quad K_\varphi \cap (-K_\varphi) = \{0\}$, *where* $K_\varphi = \{u \in U;\ \varphi(u) = \varphi(0) = 0\}$.

Assume moreover that A and B have compact resolvent and (23). Then strong asymptotic stability holds for $S^B(t)$ if and only if

(43) $\quad \forall i \in \mathbb{N},\ Ker\, C \cap F_i = \{0\}$.

Proof. We apply Theorem 4, and thus, have to compute K_ψ.

$$K_\psi \cap (-K_\psi) = \{v \in V\ ;\ Cv \in K_\varphi\} \cap \{v \in V\ ;\ C(-v) \in K_\varphi\}$$
$$= \{v \in V\ ;\ Cv \in K_\varphi \cap (-K_\varphi)\} = \mathrm{Ker}\, C \ \text{ by (42).}$$

Therefore, (43) is equivalent to (30). Clearly (19) is satisfied, and (23) is assumed. Thus Theorem 4 applies ∎

Remark 1. Now, we can discuss the meaning of Corollary 5 in a framework familiar in control theory, namely when the observation operator is the dual of the control operator. Besides (42) assume moreover that

(44) $\quad \partial\psi(v) = C^* \partial\varphi(Cv)$

which is certainly true is φ is regular or also if C is surjective (see [LA] and Appendix 1). Assume also

(45) $\quad \partial\varphi(0) \subset \mathrm{Ker}\, C^*$ (for instance $\partial\varphi(0) = \{0\}$!)

Then (23) is satisfied and Corollary 5 applies.

Remark 2. Let us note that (43) is equivalent to a rank condition. Indeed, (43) means that C restricted to F_i is injective, or else rank $C_{|F_i} = \dim F_i$. This type of condition appears naturally when one wants to characterize weak observability for the uncontrolled system

(46) $\quad \begin{cases} w_{tt} + Aw = 0 \\ z = Cw_t \end{cases}$

that is,

(47) $\quad Cw_t = 0,\ \forall\, t \geq 0 \Rightarrow w \equiv 0.$

We refer to [EL JAI-PRI] for a discussion, or [TRI, Theorem 5.5] for a theory in the case where $C \in \mathscr{L}(H,U)$. Here C may be unbounded on H.

In our framework, where A is coercive with compact resolvent, it is easy to establish the equivalence between weak observability for the uncontrolled system and the range condition (43). One just has to use the expansion (28) for $w_t(t)$, and apply C on it (see Appendix 3). In the terminology of [EL JAI-PRI], (43) means that the observation operator C on the velocity is "strategic".

Remark 3. Corollary 5 is an extension of former results found in the literature, in the following sense.

Consider an abstract evolution equation of the form

$$\text{(48)} \quad \frac{dy}{dt} + \mathcal{A}y + \mathcal{B}^*u = 0$$

where $\mathcal{A}$ generates a strongly continuous semi-group of contractions on a Hilbert space $\mathcal{H}$ and $\mathcal{B}^* \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, with $\mathcal{U}$ another Hilbert space. Assume $\mathcal{A}$ has compact resolvent.

Then following [BE], it is known that system (48) is strongly stabilizable iff the weakly (or strongly, by compactness of the resolvent) unstable states are approximatively controllable. In that case, $u = -\mathcal{B}y$ (or more generally, $u = -K\mathcal{B}y$, where $K \in \mathcal{L}(\mathcal{U})$ is coercive), with $\mathcal{B} \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ is a stabilizing feedback.

Consider now our case where

$$\text{(49)} \quad \mathcal{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} \text{ and } \mathcal{B} = [0, C], \ \mathcal{H} = V \times H, \ \mathcal{U} = U.$$

Since $\mathcal{A}$ generates a group, all states are unstable.

We recall that, roughly speaking, approximate controllability of a pair $(\mathcal{A}, \mathcal{B}^*)$ is equivalent to weak observability of the dual pair $(\mathcal{A}^*, \mathcal{B})$, and, as remarked previously, characterized by rank conditions.

Hence we have proved a <u>nonlinear unbounded</u> extension of the result of [BE], namely, in the framework of second order systems : The system

$$\text{(50)} \quad y_{tt} + Ay + C^*u = 0 \quad \text{with } C \in \mathcal{L}(V,U),$$

is strongly stabilizable iff the pair (A,C) is weakly observable. In that case $u \in \partial\varphi(Cy_t)$ is a stabilizing feedback, provided $\varphi : U \to [0,\infty]$ is convex, l.s.c., proper, and satisfies (42) and (44)-(45) (with compactness of B).

We observe that Theorem 4 goes beyond this formulation, since it does not need the introduction of any observation operator C.

Remark 4. Another interesting feature of our formulation is that it can handle "unilateral" feedback conditions, since conditions (30) or (42) concern $K\psi \cap (-K\psi)$, or $K\varphi \cap (-K\varphi)$, and not separately K_ψ or K_φ. This will be used in examples in next section.

Remark 5. Finally, we would like to remark that Theorem 2 is a way of systematically reducing the problem of stabilization to the verification of an adequate uniqueness property for the operator A, in an abstract "unbounded nonlinear" framework (for the damping term). For a similar point of view, in a linear or nonlinear framework, see [DA], [Q-R] for an abstract formulation, and [LAG$_1$] [LA$_1$] [Q-R] for applications. In particular, the formalization and results in [DA] are very similar to ours, though developed for bounded feedback.

3. <u>Applications</u>

Now we show how the strong stabilization for various nonlinear feedback terms can be deduced from Theorem 4 or Corollary 5, in the case of wave, beam or plate-like equations. Of course, with our technique, we do not obtain any estimate of the decay to zero. Other techniques are needed, together generally with geometric assumptions on the domain, see for example [CHE] [LAG] [LA] [KO-ZU] [ZU] for the wave equation with boundary damping, and [LA] [LAG_1] for plate-like equations.

3.1. Wave equation (N = 1)

Just to show how our method works, we first study the following academic and rather well-known case.

We consider a string which is clamped at the left end, and controlled at the right end by a force which is a nonlinear function of the observed transversal velocity. Our method handles as well the case of variable mass density b(x) and flexibility a(x), provided a, b $\in L^\infty(0,1)$, a, b $\geq m > 0$ (see Section 3.2 for the case of variable coefficients, in the slightly more complicated case of an Euler-Bernoulli beam equation). However, just for the sake of simplicity, we take $a \equiv b \equiv 1$. Hence we consider the system

$$(51)\quad \begin{cases} u_{tt} - u_{xx} = 0\ ,\ 0 < x < 1, & t > 0, \\ u\ (0,t) = 0\ , & t > 0, \\ -u_x(1,t) \in \alpha(u_t(1,t))\ , & t > 0, \end{cases}$$

where α is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \alpha(0)$.

We set (51) into the abstract formulation of Section 2. Let

$H = L^2(0,1)$, $V = \{v \in H^1(0,1)\ ;\ v(0) = 0\}$

$$< \tilde{A}u, v >_{V'xV} = \int_0^1 u_x\, v_x\, dx,$$

$$D(A) = \{u \in V;\ u_{xx} \in L^2(0,1)\ ;\ u_x(1) = 0\},$$

$$\forall\ u \in D(A),\ Au = -u_{xx}.$$

Let $j : \mathbb{R} \to [0,\infty]$ be convex, l.s.c., proper, such that $\partial j = \alpha$, and $j(0) = 0$. We set

$$\psi(v) = j(v(1)) = j(Cv),\ \forall\ v \in V,$$

where $C \in \mathcal{L}(V,\mathbb{R})$ is the trace operator at x = 1 (C is surjective, and thus $\partial\psi(v) = C^*\ \alpha(Cv)$). Noticing that $\partial\psi(h) = \alpha(h(1))\ \delta_1$, and that

$$< \tilde{A}v + \partial\psi(h),\ w >_{V'xV} = v_x(1)\ w(1) + \alpha(h(1))\ w(1) - \int_0^1 v_{xx} w dx$$

for regular v, choosing then $w \in V$ such that w(1) = 0, and then w arbitrarily, we get from (5)

$$D(B) = \{(v,h) \in VxH\ ;\ h \in V\ ;\ h(1) \in D(\alpha)\ ;\ -v_x(1) \in \alpha(h(1))\ ;\ v_{xx} \in H\}$$

and from (6), if $(v,h) \in D(B)$

$$B(v,h) = \{-h,\ -v_{xx}\}.$$

Since $D(B) \supset D(A) \times \{h \in V\ ;\ h(1) = 0\}$, it is obvious that D(B) is dense in VxH, and thus, (51) is well-posed on VxH. We note that A has compact resolvent, by classical regularity. The same result holds for B, since $(v,h) = (I+B)^{-1}\ (f,g)$ satisfies

$$\begin{cases} v - h = f \in V \\ h - v_{xx} = g \in H \\ (v,h) \in V\times V\ ,\ -v_x(1) \in \alpha(h(1)) \end{cases}$$

or else,

$$\begin{cases} v - v_{xx} = g + f \in H \\ v \in V\ ,\ -v_x(1) \in \alpha(v(1) - f(1)) \\ v_{xx} \in H. \end{cases}$$

Multiplying the equation by v, and then by v_{xx}, we obtain

$$\|v\|_V^2 + \|v_{xx}\|_H^2 \le C\, [\|f\|_V^2 + \|g\|_H^2]$$

(see [CO-PI] for the details in the case of the beam equation, which is in fact slightly more complicated than the present case). Clearly the estimate implies compactness. We see here that no compactness assumption on the graph α is necessary to conclude.

The eigenvalues of A are ω_k^2, where $\omega_k = (2k+1)\,\frac{\pi}{2}$, with eigenfunctions $\varphi_k(x) = \sin \omega_k x$. All eigenvalues are simple.

Now we apply theorem 4, with $\psi(v) = j(v(1))$, thus $K_\psi = \{v \in V\ ;\ v(1) \in j^{-1}(0)\}$. Condition (23) is equivalent to $\alpha(0) = \{0\}$, i.e. j is not "vertical" at 0.

Let us now study condition (30) of Theorem 4. If $j^{-1}(0)$ contains a neighborhood of 0, then, for any $v \in V$, there exists $\lambda \in \mathbb{R}$ such that $\pm\, \lambda v(1) \in j^{-1}(0)$, and thus $\lambda v \in K_\psi \cap (-K_\psi)$. So condition (30) is not true and strong stabilization does not hold. On the other hand, if $j(\xi) > 0$ for $\xi > 0$ or $\xi < 0$, then $K_\psi \cap (-K_\psi) = \{v \in V;\ v(1) = 0\}$. Thus condition (30) reduces to showing the following "uniqueness" result

$$\begin{cases} -u_{xx} = \lambda u \\ u(0) = 0\ ,\ u_x(1) = 0 \\ u(1) = 0 \end{cases} \Rightarrow\ u \equiv 0$$

which is straightforward.

So, if j is not "flat" at 0 (on both sides of 0) (for (30)) and also not vertical at 0 (for(23)), strong asymptotic stabilization holds for (51). Note that this is the case for instance with the unilateral boundary condition $-\, u_x(1,t) = [u_t(1,t)]^+$.

3.2. Euler-Bernoulli beam equation

We consider a beam which is clamped at the left end, and controlled at the right end by a force and a moment which are nonlinear functions of the transversal and angular velocities. As announced previously, we consider the case of variable mass density b and flexural rigidity a such that $a, b \in L^\infty(0,1)$, $a, b \ge m > 0$

$$(52)\quad \begin{cases} b\, u_{tt} + (au_{xx})_{xx} = 0\ ,\ 0 < x < 1\ , & t > 0 \\ u(0,t) = u_x(0,t) = 0\ , & t > 0 \\ -au_{xx}(1,t) \in \beta(u_{xt}(1,t))\ , & t > 0 \\ (au_{xx})_x\ (1,t) \in \alpha(u_t(1,t))\ , & t > 0, \end{cases}$$

where α and β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \alpha(0)$, $0 \in \beta(0)$. This problem has been specifically studied in [CO-PI]. Here we show that it fits in our general abstract framework.

Let $H = L^2(0,1)$ equipped with the scalar product $(u,v)_H = \int_0^1 b(x)\, u(x)\, v(x)\, dx$,

$V = \{v \in H^2(0,1)\ ;\ v(0) = v_x(0) = 0\}$,

$< \tilde{A}\, u,v >_{V'xV} = \int_0^1 au_{xx}\, v_{xx}\, dx$,

$D(A) = \{u \in V\ ;\ (au_{xx})_{xx} \in H\ ;\ au_{xx}(1) = (au_{xx})_x(1) = 0\ \}$

$\forall\ u \in D(A)$, $Au = \frac{1}{b}\,(au_{xx})_{xx}$.

Let $j_1, j_2 : \mathbb{R} \to [0,\infty]$ be convex, l.s.c. and proper such that $\partial j_1 = \alpha$, $\partial j_2 = \beta$, $j_1(0) = j_2(0) = 0$. We set $\psi(v) = j_1(v(1)) + j_2(v_x(1)) = \varphi(Cv)$, $\forall\ v \in V$, where

$\varphi(\xi_1, \xi_2) = j_1(\xi_1) + j_2(\xi_2)$, $\forall\ (\xi_1, \xi_2) \in U = \mathbb{R}^2$,

and $C \in \mathscr{L}(V,U)$ is defined by $Cv = \begin{pmatrix} v(1) \\ v_x(1) \end{pmatrix}$.

Then $\partial\psi(v) = \alpha(v(1))\, \delta_1 - \beta(v_x(1))\, \delta'_1$ (again, C is surjective, so that $\partial\psi(v) = C^*\partial\varphi(Cv)$). Noticing that, for regular v, one has

$<\tilde{A}\ v\ + \partial\psi(h), w>_{V'xV}\ = av_{xx}(1)w_x(1) - (av_{xx})_x(1)w(1) + \alpha(h(1))w(1) +$

$\beta(h_x(1))w_x(1) + \int_0^1 (av_{xx})_{xx}\ w\ dx,$

we get immediately from (5)

$D(B) = \{(v,h) \in VxH\ ;\ h \in V\ ;\ h(1) \in D(\alpha)\ ;\ h_x(1) \in D(\beta);\ av_{xx}(1) \in \beta(h_x(1));$
$(av_{xx})_x(1) \in \alpha(h(1));\ (av_{xx})_{xx} \in H\}$

and if $(v,h) \in D(B)$,

$B(v,h) = (-h, \frac{1}{b}\,(av_{xx})_{xx})$.

With arguments similar to those mentioned in Subsection 3, one can prove that D(B) is dense in V x H, thus (52) is well-posed on VxH, and that A and B have compact resolvent [CO-PI]. No compactness assumptions on α or β are necessary to conclude.

The eigenvalues and eigenvectors of A are given by

$$\begin{cases}(a\varphi_{xx})_{xx} = \omega^2 b\varphi \\ \varphi(0) = \varphi_x(0) = 0 \\ a\varphi_{xx}(1) = (a\varphi_{xx})_x(1) = 0.\end{cases}$$

All the eigenvalues ω_k^2, k = 1, 2,..., are simple, and $\varphi_k(1)$ and $\varphi_{kx}(1)$ are nonzero for any k [CO-PI].

Now we apply Theorem 4. Here

$K_\psi = \{v \in V\ ;\ v(1) \in j_1^{-1}(0)\ ;\ v_x(1) \in j_2^{-1}(0)\ \}.$

Condition (23) is equivalent to $\alpha(0) = \{0\}$ and $\beta\,(0) = \{0\}$, i.e. the two graphs are not "vertical" at the origin.

Let us now study condition (30) of Theorem 4. If both $j_1^{-1}(0)$ and $j_2^{-1}(0)$ contain a neighborhood of the origin, then, for any $v \in V$, there exists $\lambda \in \mathbb{R}$ such that $\pm\lambda v \in K_\psi$, thus strong stabilization is not possible. On the other hand, if $j_1(\xi) > 0$ for $\xi > 0$ or $\xi < 0$, or if j_2 has this property, then

$K_\psi \cap (-K_\psi) = \{v \in V\ ;\ v(1) = 0\}$ or $\{v \in V\ ;\ v_x(1) = 0\}.$

Thus condition (30) amounts to proving the following uniqueness result

$$(53) \qquad \begin{cases}(au_{xx})_{xx} = \lambda bu \\ u(0) = 0\ ,\ u_x(0) = 0 \\ au_{xx}(1) = 0\ ,\ (au_{xx})_x(1) = 0,\end{cases}$$

<u>and</u> $u(1) = 0$ <u>or</u> $u_x(1) = 0$, imply $u \equiv 0$.

But this is always true, as a consequence of the simplicity of the eigenvalues (see [CO-PI] for a proof).

Thus strong asymptotic stability holds if the two graphs α and β are not vertical at 0, one at least being not "flat" at 0.

Note that the result is true for instance with the following unilateral feedback

$au_{xx}(1,t) = 0,\ (au_{xx})_x(1,t) = [u_t(1,t)]^+.$

It is also possible to generalize equation (52) by taking coupled boundary conditions, that is (with $a = b \equiv 1$ for simplicity) $\psi(v) = \varphi(Cv)$ where

$\varphi : \mathbb{R}^2 \to [0,\infty]$ is convex, l.s.c. and proper

$\varphi(0) = \min \varphi$ and $Cv = \begin{pmatrix} v(1) \\ v_x(1)\end{pmatrix}$.

For instance, consider $\varphi(\xi,\eta) = \frac{1}{2}\,(\tau\xi - \eta)^2,\ \tau > 0.$

Here Corollary 5 is not applicable since $K_\varphi \cap (-K_\varphi) \neq \{0\}$. However Theorem 4 applies. Indeed,

$K_\psi = \{v \in V\ ;\ \tau\, v(1) - v_x(1) = 0\}$ and

$\langle \partial\psi(v), h \rangle_{V' \times V} = [\tau v(1) - v_x(1)]\ [\tau h(1) - h_x(1)]$

thus (23) is obviously true.

For condition (30), one has to prove the following uniqueness result : if u is a solution of (53) in the specific case $a = b \equiv 1$, and if moreover $\tau u(1) - u_x(1) = 0$, then $u \equiv 0$.

Let $u \not\equiv 0$ be a solution of (53). Then $u = \varphi_k$, $\lambda_k = \omega_k^2$, and one has to prove that

$$\tau\, \varphi_k(1) - \varphi_{kx}(1) \neq 0 \ \forall\, k, \text{ or else } \varphi_k(1)\, [\tau - \frac{\varphi_{kx}(1)}{\varphi_k(1)}] \neq 0,\ \forall\, k = 1, 2, \ldots$$

But for the normalized eigenfunctions, it is known that

$$\varphi_k(1) = 2(-1)^{k+1}, \text{ and } \lim_{k \nearrow \infty} \left|\frac{\varphi_{kx}(1)}{\varphi_k(1)}\right| = +\infty \text{ [CO]}.$$

So strong stabilization occurs for the problem

$$(54)\quad \begin{cases} u_{tt} + u_{xxxx} = 0 \\ u(0,t) = u_x(0,t) = 0 \\ u_{xx}(1,t) = \tau\, u_t(1,t) - u_{xt}(1,t) \\ u_{xxx}(1,t) = \tau\, [\tau\, u_t(1,t) - u_{xt}\,(1,t)] \end{cases}$$

for any $\tau > 0$, except for a sequence going to infinity. In particular, (54) is strongly stable for any small $\tau > 0$.

Remark. Assume a and b are piecewise regular, and for simplicity, constant on (x_{i-1}, x_i) with $x_0 = 0$, $x_N = 1$. With the previous notations, consider

$$\psi(v) = \frac{1}{2} \sum_{i=1}^{N} [\alpha_i\, v^2(x_i) + \beta_i\, v_x^2(x_i)]\, ,\ \alpha_i \geq 0,\ \beta_i \geq 0.$$

Then the abstract formulation covers the following problem (A is the same as previously)

$$b_i u_{tt} + a_i\, u_{xxxx} = 0 \quad \text{on } (x_{i-1}, x_i),\quad i = 1, \ldots N$$

$$u(0,t) = 0\, ,\ u_x\,(0,t) = 0$$

$$u(x_i-,t) = u(x_i+,t),\ u_x(x_i-,t) = u_x(x_i+,t)$$

$$\begin{pmatrix} a_i u_{xxx}(x_i-,t) - a_{i+1}\, u_{xxx}\,(x_i+,t) \\ -\, a_i u_{xx}(x_i-,t) + a_{i+1}\, u_{xx}\,(x_i+,t) \end{pmatrix} = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix} \begin{pmatrix} u_t\,(x_i,t) \\ u_{tx}\,(x_i,t) \end{pmatrix},\ i = 1, \ldots N-1$$

$$\begin{pmatrix} a_N\, u_{xxx}(1,t) \\ -a_N\, u_{xx}(1,t) \end{pmatrix} = \begin{pmatrix} \alpha_N & 0 \\ 0 & \beta_N \end{pmatrix} \begin{pmatrix} u_t(1,t) \\ u_{tx}(1,t) \end{pmatrix}.$$

This problem has been considered in [CHE-DE], where uniform exponential decay is proved under the assumption $\alpha_N > 0$, $b_i \leq b_{i+1}$, $a_i \geq a_{i+1}$, $\forall i$.

Applying our Theorem 4, we obtain the strong stability under the assumptions $\alpha_i, \beta_i \geq 0$ $i = 1, \ldots N$, and α_N or $\beta_N > 0$, as in the previous case. If $\alpha_N = \beta_N = 0$, but

α_{i_0} or $\beta_{i_0} > 0$ for some i_0, then, due to the simplicity of all eigenvalues, x_{i_0} has to be "strategic" in the sense that it has to be different from all the zeroes of the eigenfunctions or the derivatives of the eigenfunctions.

With our method, it is not hard to consider nonlinear feedback laws as previously, and even to combine this with coupled interaction between the transversal and angular velocities at the nodes x_i, $i=1,...N$.

3.3 Hybrid system (for Euler-Bernoulli beams)

We consider an homogeneous Euler-Bernoulli beam clamped at the left end, and controlled at the right end by a moment, but now, there is a mass and inertia at this end. Normalizing the constants, the model is the following

$$\begin{cases} w_{tt} + w_{xxxx} = 0 \quad 0 < x < 1, \ t > 0, \\ w(0,t) = w_x(0,t) = 0, & t > 0, \\ - w_{xx}(1,t) = w_{xtt}(1,t) + f(t), & t > 0, \\ w_{xxx}(1,t) = w_{tt}(1,t), & t > 0. \end{cases}$$

With $f(t) = w_{xt}(1,t)$ if $|w_{xt}(1,t)| \leq r$, $f(t) = r \operatorname{sgn}(w_{xt}(1,t))$ otherwise, we get the problem studied in [SLE], where strong asymptotic stability has been proved.

Let us show that we can also handle this problem with our technique. We set
$H = L^2(0,1) \times \mathbb{R} \times \mathbb{R}$,
$V = \{(w,a,b) \in H \,;\, w \in H^2(0,1) \,;\, w(0) = w_x(0) = 0 \,;\, a = w(1) \,;\, b = w_x(1)\}$,

$$\left(\tilde{A} \begin{pmatrix} w \\ a \\ b \end{pmatrix} , \begin{pmatrix} \varphi \\ \alpha \\ \beta \end{pmatrix} \right)_{V'xV} = \int_0^1 w_{xx}\varphi_{xx},$$

$D(A) = \{(w,a,b) \in V \,;\, w \in H^4(0,1)\}$,
and, if $(w,a,b) \in D(A)$,

$$A \begin{pmatrix} w \\ a \\ b \end{pmatrix} = \begin{pmatrix} w_{xxxx} \\ -w_{xxx}(1) \\ w_{xx}(1) \end{pmatrix} .$$

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$\begin{cases} \varphi(\xi) = \xi \text{ if } |\xi| \leq r \\ \varphi(\xi) = r \operatorname{sgn}(\xi) \text{ if } |\xi| \geq r \end{cases}$$

and j the primitive of φ such that $j(0) = 0$. For $(w,a,b) \in V$, we set $\psi(w,a,b) = j(b)$. Then is is not difficult to see that the abstract formulation

$$u_{tt} + \tilde{A}u + \partial\psi(u_t) = 0, \ u = (w,a,b),$$

recovers the initial problem of [SLE]. So this problem is well-posed on VxH, as a first order equation. Omitting the details concerning the precise definition of B the density of D(B) in VxH and the compactness of the resolvents (easily obtained using Appendix 2), the asymptotic stability amounts, through Theorem 4, to proving the following "uniqueness" result.

Let φ be an "eigenvalue" of A, $\varphi = (w,a,b)$, such that $\psi(\varphi) = 0$. Then $\varphi \equiv 0$, or else, if w satisfies

$$\begin{cases} w_{xxxx} = \lambda w \\ -w_{xxx}(1) = \lambda a = \lambda w(1) \\ w_{xxx}(1) = \lambda b = \lambda w_x(1) \\ w(0) = w_x(0) = 0 \end{cases}$$

and $w_x(1) = 0$

then $w \equiv 0$ (thus also $a = b = 0$). But, in [SLE] it is proved that the eigenvalues of this equation are simple and that, moreover $w_x(1) \neq 0$ (see [SLE, (8.29)] (the property to prove is rather complicated to establish). Whence the strong asymptotic stabilization.

Remark 1. Instead of controlling by a moment, one can control by a force, that is, boundary conditions are now

$$\begin{cases} -w_{xx}(1,t) = w_{xtt}(1,t) \\ w_{xxx}(1,t) = w_{tt}(1,t) + g(t), \end{cases}$$

where $g(t) = w_t(1,t)$, possibly also truncated as before.

Remark 2. One can combine the model studied in the Remark of Section 3.2 with the previous one. Then one can study the case of serially connected Euler-Bernoulli beams with, at each node (of possibly nonzero mass and nonzero inertia) control by force and/or moment, that is, at each node

$$\begin{cases} a_i\, u_{xxx}\,(x_{i-},t) - a_{i+1}\, u_{xxx}\,(x_{i+},t) = m_i\, u_{tt}\,(x_i,\, t) + g_i(t) \\ -\, a_i\, u_{xx}\,(x_{i-},t) + a_{i+1}\, u_{xx}\,(x_{i+},t) = J_i\, u_{xtt}\,(x_i,\, t) + f_i(t), \end{cases}$$

where f_i and g_i are of the form just presented (with, or without, truncation). But of course the reduced "uniqueness" result becomes rather hard to handle...

3.4. Wave equation ($N \geq 1$)

We consider the wave equation where control is exerted by means of a force which is a nonlinear function of the observed velocity, on a part Γ_0 of the boundary Γ, assumed to be regular. In the sequel (Γ_0, Γ_*) is a partition of Γ, and we assume meas $(\Gamma_0) > 0$, int $\Gamma_* \neq \emptyset$.

The system is the following

$$(55)\qquad \begin{cases} u_{tt} - \Delta u = 0,\ x \in \Omega,\ t > 0 \\ u = 0,\ x \in \Gamma_*,\ t > 0 \\ \dfrac{\partial u}{\partial \nu} = -a(x)\ g(u_t),\ x \in \Gamma_0,\ t > 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is monotone, continuous (just for simplification), such that $g(0) = 0$, $a > 0$ is continuous, and ν is the normal unit vector on Γ pointing outwards Ω.

When $\Gamma_0 = \{x \in \Gamma\ ;\ (x-x_0).\nu > 0\ \}$, $\Gamma_* = \Gamma \setminus \Gamma_0$, where $x_0 \in \mathbb{R}^N$, and $\overline{\Gamma_0} \cap \Gamma_* = \emptyset$ if $N > 3$, strong stability holds for problem (55), with estimates for the decay depending on the behaviour of g, see for instance [ZU]. Strong stability has also been proved in [LA_1] in this framework, for more general g.

Here we obtain strong stabilization for very general partitions (Γ_0, Γ_*) of the boundary. First, we set (55) in our abstract framework. We set

$g = \partial j$, $H = L^2(\Omega)$, $V = \{v \in H^1(\Omega)\,;\, v = 0 \text{ on } \Gamma_*\}$,

$$< \tilde{A}u, v >_{V'xV} = \int_\Omega \nabla u . \nabla v \, dx,$$

$$D(A) = \{u \in V;\ \Delta u \in L^2(\Omega)\,;\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$

and, if $u \in D(A)$, $Au = -\Delta u$.

Let $\psi(v) = \int_{\Gamma_0} a(x)\, j(v(x))\, d\sigma(x), \quad \forall v \in V.$

Assume g satisfies a suitable growth condition. Then $\forall\, v \in V$, $\partial\psi(v) = \{f\}$, where (see Appendix 4)

$$< f,h >_{V'xV} = \int_{\Gamma_0} a(x)\, g(v(x))\, h(x)\, d\sigma(x),$$

so that, from (5), one deduces

$$D(B) = \{(v,h) \in VxV\,;\ \Delta v \in L^2(\Omega)\,;\ \frac{\partial v}{\partial \nu} = -\,ag(h) \text{ on } \Gamma_0\}$$

and, if $(v,h) \in D(B)$, $B(v,h) = (-h,-\Delta v)$.

Since $D(B) \supset D(A) \times \{h \in V\,;\, h = 0 \text{ on } \Gamma_0\}$, it follows that D(B) is dense in VxH, so that (55) is well-posed on VxH. Compactness of A is obvious and compactness of B follows from the adequate growth condition on g, assuming Green's formula is valid (see Appendix 4).

Thus we can apply Theorem 4. We first observe that obviously (23) is satisfied (clearly, one can replace g by a maximal monotone graph α such that $\alpha(0) = \{0\}$).

Let us now show that (30) is satisfied. Assume $K_j \cap (-K_j) = \{0\}$ that is, g is not "flat" at 0, as usual. Then

$K_\psi \cap -K_\psi = \{v \in V\,;\, j(v(\sigma)) = 0\,,\, j(-v(\sigma)) = 0, \text{ a.e. on } \Gamma_0\}$

$= \{v \in V\,;\, v(\sigma) \in K_j \cap (-K_j) \text{ a.e. on } \Gamma_0\}$

$= \{v \in V\,;\, v = 0 \text{ a.e. on } \Gamma_0\}$.

So condition (30) amounts to proving the following "uniqueness" result :

$$(56)\quad \begin{cases} -\Delta\varphi = \omega^2\varphi \\ \varphi = 0 \text{ on } \Gamma_* \\ \dfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Gamma_0 \end{cases}$$

<u>and</u>

(57) $\varphi = 0$ on Γ_0

imply $\varphi \equiv 0$ in Ω.

This uniqueness result holds for very general situations where Γ_0 is not too "thin". For instance, if Γ_0 contains $B(x_0, \varepsilon) \cap \Gamma$, where $x_0 \in \Gamma$, and Γ is regular. The proof is elementary and proceeds by extending φ by 0 outside Γ (and near Γ_0), using analyticity properties. So, we have an extension of the results of [LA1].

When $\Gamma_0 = \{x \in \Gamma \,;\, (x-x_0).\nu > 0\}$, one can prove the uniqueness result by the usual multiplier's technique. However, in that case, we get much more than uniqueness (or equivalently weak observability), namely strong observability.

One can easily be convinced that stabilization holds for more general partitions by taking for instance a rectangular membrane $\Omega = (0,a) \times (0,b)$, with $u = 0$ on the vertical edges Γ_*, and $\frac{\partial u}{\partial \nu} \in -\beta(u_t)$ on the horizontal edges Γ_0, where β is a maximal monotone graph satisfying adequate growth conditions, so that the compactness assumptions hold. In that case, the uniqueness result can be proved in an elementary way, using Fourier expansions.

The solutions of the eigenvalue problem (56) are

$$\omega_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \; \varphi(x,y) = \sum_{r,s} \alpha_{rs} \sin\frac{\pi r x}{a} \cos\frac{\pi s y}{b}$$

where $r \in \mathbb{N}^*$, $s \in \mathbb{N}$, $\frac{r^2}{a^2} + \frac{s^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}$, with $m \in \mathbb{N}^*$, $n \in \mathbb{N}$.

For $y = 0$, or $y = b$, one gets $\varphi_{|\Gamma_0} = \sum_{r,s} \pm\, \alpha_{rs} \sin\frac{\pi r x}{a} = 0$. This implies $d_{rs} = 0$ for all r, s. Thus (56) and (57) imply $\varphi \equiv 0$.

Clearly, the uniqueness result holds also for any Γ_0 which contains an arbitrarily small horizontal interval.

Instead of boundary feedback, one can also consider interior feedback, in a fairly general framework. For simplicity, consider again the rectangular membrane with Dirichlet conditions on Γ_* and Γ_0 (or Neumann conditions on Γ_0).

Let $j : \mathbb{R} \to [0,\infty]$ be a convex, l.s.c. proper function such that $\min j = j(0)$, set $\beta = \partial j$, and consider

$$(58) \quad \psi(v) = \int_\Omega j(\tilde{v})\, d\mu \;,\; D(\psi) \subset V = H_0^1(\Omega),$$

where μ is a positive Radon measure on Ω, of finite energy, and $\tilde{v}$ is the quasi-continuous representative of v, with respect to the usual V- capacity.

In particular, one can consider the case where the support of μ is a "thin" set E of positive capacity (a piece of curve for instance), or a closed set E with non empty interior. With μ the length or area measure in that case, one solves in fact the formal problem

$$u_{tt} - \Delta u + \beta(u_t)\,\mu \ni 0.$$

Since μ is of finite energy, it follows that $v_n \to v$ in $V \Rightarrow \tilde{v}_n \to \tilde{v}$, μ a.e. up to a subsequence, and hence ψ is l.s.c.. Therefore, our theory applies and, provided the compactness assumptions are true, strong stability amounts to verify (23) and (30).

The usual assumption $\beta(0) = \{0\}$ ensures (23) is true. For (30), we assume $K_j \cap (-K_j) = \{0\}$. Then we have to prove the following uniqueness result :

(59) $\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \Gamma \end{cases}$

and

(60) $\tilde{u} = 0 \quad \mu$ a.e.

imply $u \equiv 0$.

For instance if μ is the length or area measure on a piece of curve or on a closed set E of non empty interior, stabilization "usually" holds. Assume as above that $\Omega = (0,a) \times (0,b)$; then $\lambda = \omega_{mn} = \pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$ and if $\varphi_{rs}(x,y) = \sin\frac{\pi r x}{a} \sin\frac{\pi s x}{b}$, the solution of (59) can be written

$$u(x,y) = \sum_{r,s} \alpha_{rs}\, \varphi_{rs}(x,y) \text{ where } \frac{r^2}{a^2} + \frac{s^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}.$$

If $E \supset]\alpha,\beta[\times \{y_o\}$, with $\frac{y_0}{b} \notin \mathbb{Q}$, or if $E \supset \{x_o\} \times]\alpha,\beta[$ with $\frac{x_0}{a} \notin \mathbb{Q}$, then (60) implies $\alpha_{rs} = 0$ for all r,s and $u \equiv 0$. Hence (30) is true.

If the above conditions on E are not true, it may happen that a solution u of (59) is identically 0 on E, so stabilization does not hold. We observe that if E contains an open set, strong stabilization holds.

3.5. Rectangular Kirchhoff plates.

We consider a simply supported rectangular plate $\Omega = (0,a) \times (0,b)$. We can apply our general formalism to the equation

$$u_{tt} + \Delta^2 u + \partial\psi(u_t) = 0 \text{ in } \Omega,\ u = \frac{\partial^2 u}{\partial n^2} = 0 \text{ on } \partial\Omega$$

to obtain stabilization results. Moreover, one can consider functionals of the form (58) where $D(\psi) \subset H^2(\Omega)$, the capacity being now the one associated with the H^2 - norm.

In that way, one can choose Dirac masses for μ, and thus consider inner point control. Let us study one particular case, where the external force acting on the plate is exerted at the points $(p_i, q_i) \in \Omega$. This problem has been studied in [Y]. Normalizing the constants, we get the following evolutionary system

$$
(61)\quad \begin{cases} u_{tt} + \Delta^2 u = \displaystyle\sum_{i=1}^{l} f_i(t)\, \delta(x-p_i,\, y-q_i),\ x \in \Omega,\ t > 0, \\ u = \dfrac{\partial^2 u}{\partial n^2} = 0 \text{ on } \partial\Omega = \Gamma,\ t > 0, \\ u(x,y,0) = u_0(x,y), \\ u_t(x,y,0) = u_1(x,y), \end{cases}
$$

where $\Delta^2 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$, $f_i \in L^1_{loc}(\mathbb{R}_+, \mathbb{R})$. We set

$$H = L^2(\Omega),\ V = \{v \in H^2(\Omega)\ ;\ v = 0 \text{ on } \partial\Omega\}.$$

We consider the control operator $C^* = \{\delta(x-p_i, y-q_i)\}_{i=1,\dots l} \in \mathcal{L}(R^l, V')$.

In [Y], problem (61) has been considered on the state space V', with Δ^2 as an operator defined on V', with domain V, so that control becomes "distributed". The main result of [Y] can be summarized as follows :

. if $\dfrac{a^2}{b^2} \in \mathbb{Q}$, then (61) is neither approximatively controllable, nor strongly (or weakly) stabilizable, by any bounded linear feedback $\in \mathcal{L}(V, \mathbb{R}^l)$, acting on velocity,

. if $\dfrac{a^2}{b^2} \notin \mathbb{Q}$, then (61) is approximatively controllable and strongly stabilizable by the feedback $f(t) = -Cu_t$, if and only if the following (rank) condition is satisfied.

$$\sum_{j=1}^{l} \frac{1}{\lambda_{mn}\, |e_{mn}|_H}\, |\, e_{mn}(p_j, q_j)\, | \neq 0,\ \forall m,n \geq 1, \text{ i.e.}$$

$e_{mn}(p_j,q_j) \neq 0$ for some j, for any $m,n \geq 1$.

Here λ_{mn} are the eigenvalues of the free vibrating plate, and e_{mn} are the corresponding eigenmodes (the eigenspaces are one-dimensional in the case considered, and for general a and b, approximate controllability is characterized by a rank condition, see [Y, Lemma 8]). Moreover, in the particular case of one actuator ($l = 1$, $\dfrac{a^2}{b^2} \notin \mathbb{Q}$), (61) is approximatively controllable iff $\dfrac{p}{a} \notin \mathbb{Q}$ and $\dfrac{q}{b} \notin \mathbb{Q}$, and $f(t) = -u_t(t,p,q)$ is a stabilizing feedback.

The proofs of the necessary and sufficient conditions given in [Y] are based on theorems in number theory and Diophantine equations.

Here we just show that the feedback $f(t) = -Cu_t$ is stabilizing, by means of our abstract results, where the state space is $H = L^2(\Omega)$, so that the control operator C^* is unbounded. For simplicity, let us also consider the case of one pointwise actuator. We set

$$< \tilde{A}u, v >_{V' \times V} = \int_{\Omega} \Delta u\, \Delta v\, dx,\ \forall u,v \in V,$$

so that $D(A) = \{u \in V;\ \Delta^2 u \in H;\ \dfrac{\partial^2 u}{\partial n^2} = 0\}$,

$\forall$ u $\in$ D(A), Au = Δ^2u.

Let $\psi : V \to \mathbb{R}_+$ be defined by $\psi(v) = \frac{1}{2} v^2(p,q)$, so that ψ is convex (regular), and defined on the whole space V, and

$$\partial\psi(v) = v(p,q)\ \delta(\circ - p, \circ - q).$$

Hence, we study the following equation (61) with $f(t) = -u_t(t,p,q)$

(62) $\quad u_{tt} + \tilde{A}u + \partial\psi(u_t) = 0.$

For regular v, we get immediately, $\forall h \in V$, $\forall \varphi \in V$

$$< \tilde{A}v + \partial\psi(h), \varphi >_{V'xV} = \int_\Gamma \Delta u \frac{\partial\varphi}{\partial n} + \int_\Omega \Delta^2 u\ \varphi + h(p,q)\ \varphi(p,q).$$

Choosing first $\varphi \in \mathscr{D}(\Omega)$, then $\varphi \in V$ arbitrary, we get from (5)

$$D(B) = \{(v,h) \in V \times V\ ;\ \Delta^2 u + h(p,q)\ \delta(\circ - p, \circ - q) \in H\ ;\ \frac{\partial^2 u}{\partial n^2} = 0 \text{ on } \Gamma\}$$

and by (6), if (v,h) $\in$ D(B),

$$B(v,h) = \{-h, \Delta^2 u + h(p,q)\ \delta(\circ - p, \circ - q)\}.$$

Thus, formally, (61) and (62) are equivalent. We observe that $\Delta^2 u \in L^2(\Omega)$ in $\Omega \backslash B(p,q;\varepsilon)$, so that $u \in H^4(\Omega \backslash B(p,q;\varepsilon))$, and the trace $\Delta u = \partial^2 u/\partial n^2$ on $\partial\Omega$ makes sense.

We also observe that

$$\{(v,h)\ ;\ v \in H^4(\Omega) \cap V\ ;\ h \in V\ ;\ h(p,q) = 0\} \subset D(B),$$

so that D(B) is dense in V x H.

Also, A has compact resolvent in H by classical regularity (see also [Y], Lemma 3). Moreover $\partial\psi : V \to V'$ is obviously compact since Range $(\partial\psi)$ is one dimensional. Using Appendix 2, it follows that B has also compact resolvent and so our whole theory on stabilization is valid.

Now we apply Theorem 4. Here $K\psi = \{v \in V\ ;\ v(p,q) = 0\}$. Since $\partial\psi(v) = v(p,q)\ \delta(\circ - p, \circ - q) = 0$, condition (23) is satisfied.

For condition (30), one has to prove that for any eigenfunction φ, one has $\varphi(p,q) \neq 0$, where $\varphi(x,y) = \sum_{r,s} \alpha_{rs} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$, the eigenvalue being

$$\omega_{mn}^4 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \pi^4, \text{ and } \frac{r^2}{a^2} + \frac{s^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}.$$

If $\frac{a^2}{b^2} \notin \mathbb{Q}$, the eigenvalues are all simple (by contradiction) and clearly $\sin \frac{m\pi p}{a} \sin \frac{n\pi q}{b} \neq 0$ if and only if $\frac{p}{a}$ and $\frac{q}{b} \notin \mathbb{Q}$.

Hence, in case $\frac{a^2}{b^2} \notin \mathbb{Q}$, (61) is strongly stabilizable with the feedback $f(t) = -Cu_t$, iff $\frac{p}{a}$ and $\frac{q}{b} \notin \mathbb{Q}$.

Remark 1. The same result is true with $\psi(v) = \varphi(v(p,q))$, for any strictly convex regular φ. In that case, one controls with $f(t) = \varphi'(u_t(p,q,t))$, with the assumptions $\frac{a^2}{b^2}$, $\frac{p}{a}, \frac{q}{b} \notin \mathbb{Q}$.

Remark 2. One can consider (61) as a problem with boundary feedback at $(p,q) \in \partial(\Omega \setminus (p,q))$. This shows that one can stabilize a plate with a pointwise feedback (on the boundary, if we want). This result is in contrast with the case of membranes, where points have $V = H^1$ - capacity zero (here the $V = H^2$ - capacity of a point is $\neq 0$). In the case of membranes, control on a regular set E means that E must contain at least a line, roughly speaking, which is of positive capacity.

Appendix 1

Here we prove, in a slightly simpler way than in [LA], that when $\psi(v) = \phi(Cv)$, $C \in \mathscr{L}(V, U)$, $\phi : U \to]-\infty,\infty]$ convex, l.s.c. and proper, then $\partial\psi = C^*\partial\phi C(.)$ when C is surjective.

(i) let $y \in C^*\partial\phi(Cv_0)$; then $y = C^*z$, $z \in \partial\phi(Cv_0)$
$\Rightarrow \phi(w) - \phi(Cv_0) \geq \langle w - Cv_0, z \rangle_{U \times U'}$, $\forall\, w \in U$,

and with $w = Cv$, we get

$\psi(v) - \psi(v_0) \geq \langle C(v-v_0), z \rangle_{U \times U'} = \langle v - v_0, C^*z \rangle_{V \times V'}$
$= \langle v - v_0, y \rangle_{V \times V'}$, $\forall\, v \in V \Rightarrow y \in \partial\psi(v_0)$
$\Rightarrow C^* \partial\phi(Cv_0) \subset \partial\psi(v_0)$ (without using the surjectivity of C, contrary to [LA]).

(ii) we prove that $\partial\psi(v_0) \subset C^*\partial\phi(Cv_0)$, $\forall\, v_0 \in V$.

<u>Lemma</u> (as in [LA]). If $y \in \partial\psi(v_0)$, $\exists\, z \in U'$ such that $y = C^* z$.

We admit the lemma for a moment. Let $y \in \partial\psi(v_0)$. Then
$\psi(v) - \psi(v_0) \geq \langle v - v_0, y \rangle_{V \times V'} = \langle v - v_0, C^*z \rangle_{V \times V'}$ with $z \in U'$ (lemma) $= \langle Cv - Cv_0, z \rangle_{U \times U'}$ i.e. $\phi(Cv) - \phi(Cv_0) \geq \langle Cv - Cv_0, z \rangle_{U \times U'}$ $\forall\, v \in V$
and since $C : V \to U$ is surjective
$\phi(h) - \phi(Cv_0) \geq \langle h - Cv_0, z \rangle_{U \times U'}$, $\forall\, h \in U$,
thus $z \in \partial\phi(Cv_0)$ and $y = C^*z \in C^*\partial\phi(Cv_0)$ ■

Finally, we prove the lemma (more directly than in [LA]).
Let $y \in \partial\psi(v)$. Then , $\forall\, h \in V$
$\psi(v+h) - \psi(v) \geq \langle y, h \rangle_{V' \times V}$, i.e.

$< y,h >_{V'xV} \leq \phi(Cv + Ch) - \phi(Cv).$

If $h \in$ Ker C, then $<y,h >_{V'xV} \leq 0$ and thus (taking - h), $< y,h >_{V'xV} = 0$, $\forall$ $h \in$ Ker C.

Thus $y \in$ (Ker C)$^{\perp}$ = Im C* (we recall that more generally, for an unbounded operator C from E to F, closed, with dense domain, R(C) is closed $\Leftrightarrow$ by [BRE_2] (Ker C)$^{\perp}$ = Im C* ; here C is surjective).

Remark. It is suspected that the surjectivity of C is not the optimal assumption. For instance, when V and U are finite dimensional, it is known that $\partial(\phi o C) = C^*\partial\phi C$, as soon as the relative interior of $D(\phi)$ intersects the range of C [RO].

Appendix 2

On the compactness of the resolvent of B in V x H.

Essentially, in an abstract setting, one has to prove that

$$(A_1) \quad \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \text{ bounded in } V \times H \\ (u,v) \in D(B) \end{cases}$$

implies (u,v) is relatively compact in V x H.

As in the proof of Proposition 1, (A_1) is equivalent to

(A_2) $u - v = F$ bounded in V

(A_3) $v + \tilde{A}u + f = G$ bounded in H , $f \in \partial\psi(v)$

or else

(A_2) $u - v = F$

(A_3) $v + \tilde{A}v + \partial\psi(v) \ni G - \tilde{A}F$.

We recall that the unique solution v of (A_3) (see Proposition 1, proof) minimizes the functional

$$\frac{1}{2}(\theta,\theta)_{HxH} + \frac{1}{2} < \theta,\theta >_{VxV} + \psi(\theta) - < G - \tilde{A}F,\theta >_{V'xV} ,$$

and thus

(A_4) $(v,v)_{HxH} + < v,v >_{VxV} \leq < G - \tilde{A}F, v >_{V'xV} + \psi(0) - \psi(v).$

Since $\tilde{A} \in \mathscr{L}(V,V')$, $G - \tilde{A}F$ remains in a bounded set of V', and since $\psi(v)$ is bounded below by an affine function (or $\psi(0) - \psi(v) \leq 0$ if $0 \in \partial\psi(0)$), we deduce from (A_4) that

$v \in$ bounded set of V, and hence also

$u \in$ bounded set of V (by (A_2)).

But now, (A_3) implies also that

$G - v - \tilde{A}u \in \partial\psi(v)$, and since $\tilde{A}$ is coercive

$-u + \tilde{A}^{-1}(G-v) \in \tilde{A}^{-1}\partial\psi(v)$,

that is, since $G-v \in H$

$-u + A^{-1}(G-v) \in \tilde{A}^{-1}\partial\psi(v)$.

Assume now that $\partial\psi : V \to V'$ is compact, and that A^{-1} is compact in H, or else, the resolvent of A is compact in H (thus $V \hookrightarrow H$ and $D(A) \hookrightarrow V$ compactly).

Then $\tilde{A}^{-1}\partial\psi(v) \in$ compact set of V and $A^{-1}(G-v) \in$ bounded set of D(A), hence compact in V.
So $u \in$ compact set of V, $v \in$ compact set of H. We have proved that if $\partial\psi : V \to V'$ is compact, and $(I + A)^{-1} \in \mathscr{L}(H)$ is compact, then $(I + \lambda B)^{-1}$ is compact.

This is, in some sense, an extension of Theorem 2.3 in [LA]. Again, let us emphasize that in specific applications, it could be as efficient to prove the compactness of the resolvent of B directly.

Appendix 3

Proposition . *With the notations of Section 2, assume that the resolvent of A is compact. Let $(w_0, w_1) \in D(B_0)$, $(w(t), w_t(t)) = S^{B_0}(t)\,(w_0, w_1)$, and let $C \in \mathscr{L}(V,U)$, where U is a Hilbert space. Then the following two conditions are equivalent :*

(i) $Cw_t = 0$ *a.e.* $t \geq 0 \Rightarrow w \equiv 0$ *(thus* $(w_0, w_1) = 0$ *: "uniqueness")*
(ii) *rank* $C_{/F_i} = p_i = \dim F_i$, $i = 1, 2, \ldots$

Proof. We recall the expansion of $w_t(t)$, given by (28) :

$$(A_1) \quad w_t(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{p_i} [b_i^j \cos \omega_i t - a_i^j \sin \omega_i t]\, \varphi_i^j,$$

where the series converges in V, uniformly in t. Since $C \in \mathscr{L}(V,U)$, we deduce from (A_1)

$$(A_2) \quad Cw_t(t) = \sum_{i=1}^{\infty} [\left(\sum_{j=1}^{p_i} b_i^j\, C\varphi_i^j\right) \cos \omega_i t - \left(\sum_{j=1}^{p_i} a_i^j\, C\varphi_i^j\right) \sin \omega_i t]$$

or else

$$(A_2) \quad Cw_t(t) = \sum_{i=1}^{\infty} [\,\tilde{b}_i \cos \omega_i t - \tilde{a}_i \sin \omega_i t]$$

with $\tilde{a}_i, \tilde{b}_i \in U$, and the series converges in U, uniformly with respect to t.
Now the proof is easy.

(ii) $\Rightarrow$ (i). Assume $Cw_t(t) = 0$, a.e. $t \geq 0$.

By a standard argument in the theory of almost periodic functions (as in Section 2, proof of Theorem 4, but more directly), we deduce that $\tilde{a}_i = \tilde{b}_i = 0$, $i = 1, 2, \ldots$ [LE-ZH]. Thus

$$(A_3) \quad \sum_{j=1}^{p_i} b_i^j \, C\varphi_i^j = 0$$

$$(A_4) \quad \sum_{j=1}^{p_i} a_i^j \, C\varphi_i^j = 0$$

By the rank condition (ii), the vectors $C\varphi_i^j$ are linearly independent in U, so that (A_3) and (A_4) imply $a_i^j = b_i^j = 0$, $\forall\, j = 1,\ldots, p_i$, $\forall\, i = 1, 2, \ldots$, and thus $w(t) = w_t(t) = 0$, $\forall\, t \geq 0$.

(i) $\Rightarrow$ (ii). Suppose (ii) is not true for some i.

We construct $(w_0, w_1) \neq 0$, $(w_0, w_1) \in D(B)$, such that $Cw_t(t) = 0$ a.e. $t \geq 0$, and hence (i) is not true.

Assume rank $C_{|F_i} < p_i$. Then there exist $(\alpha_i^j)_{j=1,\ldots p_i} \not\equiv 0$ such that

$$\sum_{j=1}^{p_i} \alpha_i^j \, C\varphi_i^j = 0, \text{ but } \sum_{j=1}^{p_i} \alpha_i^j \, \varphi_i^j \neq 0.$$

$$\text{Let } w_0 = \sum_{j=1}^{p_i} \alpha_i^j \, \varphi_i^j, \text{ and } w_1 = 0.$$

Then $a_i^j = \omega_i \, \alpha_i^j$, $b_i^j = 0$, so that $(w(t), w_t(t)) = S^{B_0}(t)\,(w_0,w_1)$ satisfies

$$w(t) = \sum_{j=1}^{p_i} \alpha_i^j \cos \omega_i t \;\; \varphi_i^j \not\equiv 0 \text{ , but}$$

$$Cw_t(t) = -\sum_{j=1}^{p_i} \omega_i \, \alpha_i^j \sin \omega_i t \;\; C\varphi_i^j \equiv 0 \qquad \blacksquare$$

Appendix 4

Here we compute the subdifferential of ψ and prove the compactness of the resolvent of B, for the example (55) of Section 3.4.

Computation of $\partial\psi$. Let $p = \frac{2(n-1)}{n}$, $q = 2\,\frac{n-1}{n-2}$ and assume $|g(\xi)| \leq A + B\,|\xi|^r$, with $r \leq \frac{n}{n-2}$ if $n \geq 3$, $\forall r$ if $n = 2$ (and no condition if $n = 1$).

It is known that $v \in H^1(\Omega) \to v_{|\Gamma} \in L^q(\Gamma)$ is continuous. Since $g : L^q(\Gamma) \to L^p(\Gamma)$ is continuous and bounded as a Nemytskii operator, it follows that $v \in V \to g(v_{|\Gamma})$ is continuous and bounded.

Thus, if $v,h \in V$, $g(v_{|\Gamma_0}) \in L^p(\Gamma_0)$, $h \in L^q(\Gamma_0)$ and

$$\left| \int_{\Gamma_0} g(v(x))\, h(x)\, d\sigma(x) \right| \leq C^t\, |g(v)|_{L^p(\Gamma_0)}\, |h|_V.$$

Therefore for some $\theta : \Gamma_0 \to (0,1)$,

$$\psi(v+h) - \psi(v) = \int_{\Gamma_0} a(x)\, [g(v+\theta h(x)) - g(v(x))]\, h(x)\, d\sigma(x) + \int_{\Gamma_0} a(x) g(v(x))\, h(x) d\sigma(x),$$

with a first term which is bounded by $C^t\, |g(v + \theta h) - g(v)|_{L^p(\Gamma_0)}\, |h|_V$, and, since $v \to g(v)$ is continuous from V to $L^p(\Gamma_0)$, the first term is $|h|_V\, \varepsilon(|h|_V)$. This proves that with the growth condition, $\partial\psi(v) = \{f\}$, where

$$< f,h >_{V'xV} = \int_{\Gamma_0} a(x)\, g(v(x)\, h(x)\, d\sigma(x)).$$

Let us now prove the <u>compactness of the resolvent of B</u>. Suppose $(I + B)(v,h) = (F,G) \in$ bounded set in V x H, i.e.

(A_1) $\quad v - h = F$ bounded in V

(A_2) $\quad h - \Delta v = G$ bounded in H, $\quad (v,h) \in V \times V$

(A_3) $\quad \dfrac{\partial v}{\partial \nu} + a\, g(h) = 0$ on Γ_0.

It is enough to prove that $v \in$ compact set in V, since then $(A_1) \Rightarrow h$ is bounded in V, hence compact in H.

We observe that $(A_1) + (A_2) \Rightarrow - \Delta v + v = F + G$.

We multiply by v and integrate over Ω (assuming Green's formula holds)

$$\int_\Omega v^2 + \int_\Omega |\nabla v|^2 + \int_{\Gamma_0} a\, g(v-F)\, v\, d\sigma = \int_\Omega (F+G)\, v \;\Rightarrow$$

$$|v|_V^2 + \int_{\Gamma_0} a\, g(v-F)(v-F)\, d\sigma = \int_\Omega (F+G)v + \int_{\Gamma_0} a\, g(v-F)\, F\, d\sigma \Rightarrow$$

$$\begin{aligned} |v|_V^2 &\leq |F+G|_H\, |v|_V + C^t |g(v-F)|_{L^p}\, |F|_{L^q} \\ &\leq C^t\, |v|_V + C^t + C^t \left(\int |v - F|^q \right)^{1/p} \\ &\leq C^t\, |v|_V + C^t + C^t\, |v-F|_q^r \\ &\leq C^t\, [1 + |v|_V + |v|_V^r]. \end{aligned}$$

If we multiply now the equation by $-\Delta v$, we get, in a similar way

$$|v|_V^2 + |\Delta v|_H^2 \leq |F+G|_H \, |\Delta v|_H + C^t \, [1 + |v|_V^r].$$

From the two inequalities we obtain

$$|v|_V^2 + |\Delta v|_H^2 \leq C^t,$$

at least if we choose $r \leq 2$. Whence the compactness (perhaps the last restriction on r is unnecessary).

Possibly, compactness can be obtained by a more direct proof using $H^{3/2}$ estimates for the solution v (see also Theorem 2 and Corollary 2 in [LA_1]).

REFERENCES

[BA-PRE] V. BARBU, Th. PRECUPANU, Convexity and Optimization in Banach Spaces, D. Reidel Pub. Co., Dordrecht (1986).

[BE] C. D. BENCHIMOL, A note on weak stabilizability of contraction semi-groups, SIAM J. Contr. Opt. 16, 3 (1978) pp. 373-379.

[BRE] H. BREZIS, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland (1973).

[BRE_2] H. BREZIS, Analyse Fonctionnelle, Théorie et Applications, Masson (1983).

[CHE] G. CHEN, Energy decay estimates and exact boundary controllability for the wave equation in a bounded domain, J. Math. Pures Appl. 58 (1979) pp. 249-274.

[CHE-DE] G. CHEN, M. DELFOUR, A. M. KRALL, G. PAYRE, Modelling, stabilization and control of serially connected beams, SIAM J. Contr. Opt. 25 (1987) pp. 526-546.

[CO] F. CONRAD, Stabilization of beams by pointwise feedback control, SIAM J. Contr. Opt. 28, 2 (1990) pp. 423-437.

[CO-PI] F. CONRAD, M. PIERRE, Stabilization of Euler-Bernoulli beam by nonlinear boundary feedback, INRIA Report RR # 1235 (1990).

[DA] C. M. DAFERMOS, Application of the invariance principle for compact processes, J. Diff. Equs. 9 (1971) pp. 291-299.

[DA-SLE] C. M. DAFERMOS, M. SLEMROD, Asymptotic behaviour of nonlinear contraction semi-groups, J. Funct. Anal. 13 (1973) pp. 97-106.

[EL JAI-PRI] A. EL JAI, T. PRITCHARD, Sensors and actuators in distributed systems, Int. J. Control 46, 4 (1987) pp. 1139-1153.

[KO-ZU] V. KOMORNIK, E. ZUAZUA, A direct method for the boundary stabilization of the wave equation, J. Math. pures et Appl. 69 (1990) pp. 33-54.

[LA] I. LASIECKA, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary, J. Diff. Equs. 79, 2 (1989) pp. 340-381.

[LA_1] I. LASIECKA, Asymptotic behaviour of the solution to wave equation with nonlinear damping on the boundary, 8^{th} Int. Conf. Analysis and Optimization of Systems, Antibes, Lect. Notes in Contr. and Inf. Sci. # 111, A. Bensoussan, J. L. Lions Eds. (1988) pp. 472-483.

[LAG] J. E. LAGNESE, Decay of solutions of the wave equation in a bounded region with boundary dissipation, J. Diff. Equs. 50 (1983) pp. 163-182.

[LAG_1] J. E. LAGNESE, Boundary stabilization of thin plates, SIAM Studies in Applied Mathematics (1989).

[LE-ZH] B. M. LEVITAN, V.V. ZHIKOV, Almost periodic functions and differential equations, Cambridge Univ. Press (1982).

[Q-R] J. P. QUINN, D. L. RUSSELL, Asymptotic stability and energy decay rates for solution of hyperbolic equations with boundary damping, Proc. Roy. Soc. Edinburgh Sect. A, 77 (1977) pp. 97-127.

[RO] R. T. ROCKAFELLAR, Convex Analysis, Princeton Univ. Press (1972).

[SLE] M. SLEMROD, Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control, Math. Control Signals Systems (1988) pp. 265-285.

[TRI] R. TRIGGIANI, Extension of rank conditions for controllability and observability to Banach spaces and unbounded operators, SIAM J. Contr. Opt. 14, 2 (1976) pp. 313-338.

[Y] YUNCHEN YOU, Controllability and stabilizability of vibrating simply supported plate with pointwise control, Adv. in Appl. Math. 10 (1989) pp. 324-343.

[ZU] E. ZUAZUA, Uniform stabilization of the wave equation by nonlinear boundary feedback, SIAM J. Contr. Opt. 28, 2 (1990) pp. 466-477.

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